Static vs dynamically induced ensembles of density matrices

Carlos Pineda[♭], **Thomas Seligman**[♯],
Satya N. Majumdar[♮]

[♭]Instituto de Física, UNAM
[♯] Centro Internacional de Ciencias; Instituto de Ciencias Físicas, UNAM
[♮] Laboratoire de Physique Théorique et Modèles Statistiques, Université Paris-Sud

The goal

- Understand the relation between two totally different (in principle) kind of ensembles.

The goal

- Understand the relation between two totally different (in principle) kind of ensembles.

  One of this ensembles comes from purely *statical* considerations, and the other from *dynamical* ones.

  This is done at fixed “decoherence”.

The goal

- Understand the relation between two totally different (in principle) kind of ensembles.

  One of this ensembles comes from purely *statical* considerations, and the other from *dynamical* ones.

  This is done at fixed “decoherence”.

- Understand the reach of the particular results explained here, when we study other quantities, not only purity.

The statical ensemble

Drawing a random state

Pure:

Let us draw a random state

$$|\psi\rangle \in \mathcal{H}$$

randomly. For that, we have a tool: the *Haar measure* associated with the unitary group acting on $\mathcal{H}$. This allows us to choose it in a democratic way.

Drawing a random state

Pure:
Let us draw a random state

$$|\psi\rangle \in \mathcal{H}$$

randomly. For that, we have a tool: the *Haar measure* associated with the unitary group acting on $\mathcal{H}$. This allows us to choose it in a democratic way.

Mixed:
Now let us draw a random density matrix $\rho \geq 0$, trρ1, i.e.

$$\rho \in \mathcal{B}(\mathcal{H})$$

How? The unitary group does not allow us to cover all density matrices.

An idea to sample the set of density matrices!

Consider an extended Hilbert space

$$\mathcal{H} = \mathcal{H}_{\text{central}} \otimes \mathcal{H}_{\text{env}}$$

and chose randomly a state $|\psi\rangle \in \mathcal{H}$. Then, take the partial trace:

$$\rho = \text{tr}_{\text{env}} |\psi\rangle\langle\psi|.$$

An idea to sample the set of density matrices!

Consider an extended Hilbert space

$$\mathcal{H} = \mathcal{H}_{\text{central}} \otimes \mathcal{H}_{\text{env}}$$

and chose randomly a state $|\psi\rangle \in \mathcal{H}$. Then, take the partial trace:

$$\rho = \text{tr}_{\text{env}} |\psi\rangle\langle\psi|.$$

Good things:

- For $\dim \mathcal{H}_{\text{env}} \geq \dim \mathcal{H}_{\text{central}}$, we cover completely the space of density matrices.

Bad things:

- The ensemble, however, depends on $N = \dim \mathcal{H}_{\text{env}}$.
- It has a “bad” limit: $\lim_{N\to\infty} = \delta(\rho - \mathbb{1}/N)$.

The Wishart ensemble

Consider an $n \times m$ matrix X of random Gaussian entries. The central object shall be the ensemble induced by

$$X^\dagger X.$$

This ensemble of positive matrices is the Wishart ensemble. If we divide it by its trace, we have *exactly* the same ensemble as the construction using random states and tracing out.

For the normalized Wishart ensemble, we know

- The density of states
- Recently a lot of extreme values
- Its distribution of eigenvalues
- Other interesting details

$$\mathcal{P}(\lambda) \propto \delta\left(\sum_i \lambda_i - 1\right) \times \prod_i \lambda_i^{|m-n|} \prod_{i<j} (\lambda_i - \lambda_j)^2$$

A foliation of such set

We are going to study this ensemble at fixed purity

$$P(\rho) = \mathrm{tr}\rho^2.$$

This leads to a distribution of eigenvalues

$$\mathcal{P}(\vec{\lambda}) \propto \delta\left(\sum_i \lambda_i - 1\right) \delta\left(\sum_i \lambda_i^2 - P\right) \prod_i \lambda_i^{|m-n|} \prod_{i<j} (\lambda_i - \lambda_j)^2.$$

A foliation of such set

We are going to study this ensemble at fixed purity

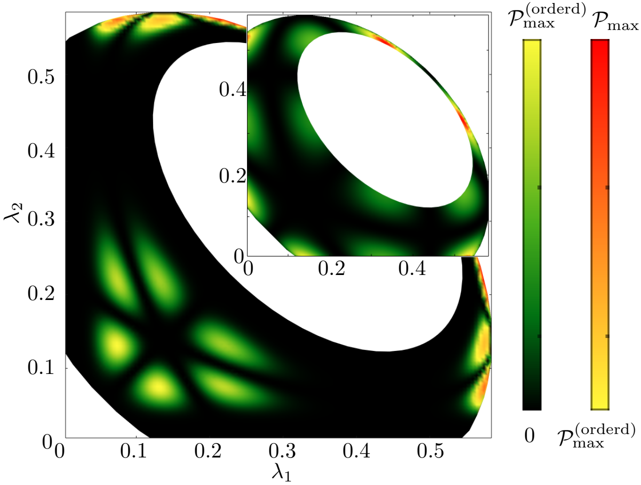
$$P(\rho) = \mathrm{tr}\rho^2.$$

This leads to a distribution of eigenvalues

$$\mathcal{P}(\vec{\lambda}) \propto \delta\left(\sum_i \lambda_i - 1\right)\delta\left(\sum_i \lambda_i^2 - P\right)\prod_i \lambda_i^{|m-n|}\prod_{i<j}(\lambda_i - \lambda_j)^2.$$

A first task

Understand in the general case, such picture



Distribution of 2 eigenvalues for the ensemble at $P = 0.4$.

A change of variables

In order to study its properties, we study only the $n-2$ free "parameters" $\lambda_1, \ldots, \lambda_{n-2}$, as the other two eigenvalues are fixed by normalization and purity conditions.

A change of variables

In order to study its properties, we study only the $n - 2$ free "parameters" $\lambda_1, \ldots, \lambda_{n-2}$, as the other two eigenvalues are fixed by normalization and purity conditions.

Then,

$$\lambda_{n-1}, \lambda_n = \frac{1 - s_1 \pm J}{2}$$

with

$$J = \sqrt{2(P - s_2) - (1 - s_1)^2}$$

being proportional to the Jacobian of the "change of variables" and $s_k = \sum_{i=1}^{n-2} \lambda_i^k$.

The distribution, reloaded

The distribution can be written in the new variables as

$$\mathcal{P}(\vec{\Lambda})\mathrm{d}\Lambda \propto J(\lambda, P)\ g^{|m-n|}(\lambda)\ V(\lambda)\ \mathrm{d}\lambda$$

Where

- $J(\lambda, P)$ is the Jacobian of the transformation,
- $g(\lambda) = \prod_i \lambda_i$ is "the peak", and
- $V(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2$ is a Vandermonde determinant.

The peak

Here we will study the function g:

$$g(\lambda) = \prod_i \lambda_i = \frac{s_2 + (s_1)^2 - 2s_1 + 1 - P}{2} \prod_{i=1}^{n-2} \lambda_i$$

has a maximum at $\lambda_{1,2,\dots,n-2} = \Lambda_\pm$. We approximate g with a parabola, and, since $(1 - x^2)^m \approx \mathrm{e}^{-mx^2}$ We obtain

$$g(\lambda_i, P)^{|m-n|} \approx g_{\mathrm{max}}^{|m-n|} \exp^{-\frac{\tilde{\lambda}^T \cdot A \cdot \tilde{\lambda}}{\sigma^2_{m,P}}}, \quad \sigma^2_{m,P} = \frac{g_{\mathrm{max}}}{\alpha(m-n)}.$$

- There is a peak at Werner-like states.
- It is Gaussian with width $\propto 1/\sqrt{m}$.

The nodes

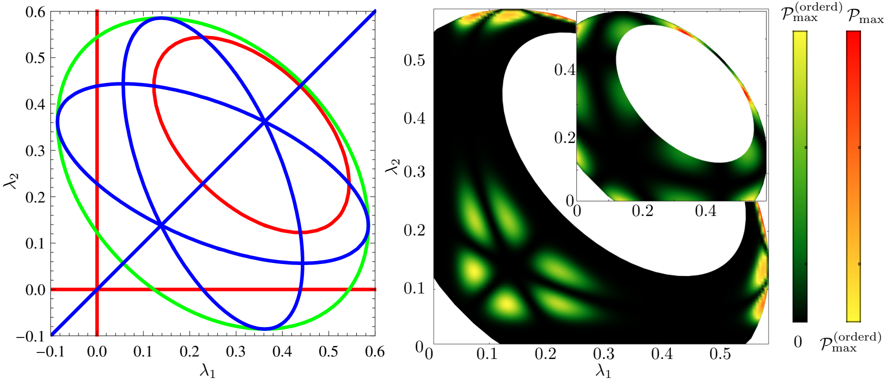
Let us study the term $V(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2$. The conditions for $\prod_{i<j}(\lambda_i - \lambda_j) = 0$ translate into

- simple hyperplanes for the case $\lambda_i = \lambda_j$, $i, j \leq n - 2$,
- an ellipsoid $Q_j(\lambda_i) = 0$ for $\lambda_i = \lambda_{n-1}$ or λ_n,
- and the condition $\lambda_{n-1} = \lambda_n$ that leads to $Q_n = 0$.

The limiting condition $\lambda_i = 0$, $i \leq n - 2$, leads to simple hyperplanes, whereas $\lambda_{n-1} = 0$ results in the ellipsoid

- $Q_{n-1} = (P - s_2) - (1 - s_1)^2 = 0$.

The $n = 4$ case



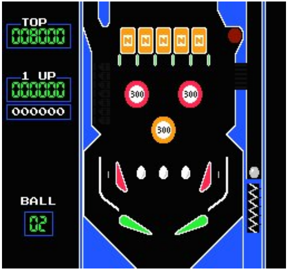
- We can clearly see the nodes, and the limiting regions
- We see a main peek,
- We observe a qualitative change for $m = 4$

The dynamical ensemble

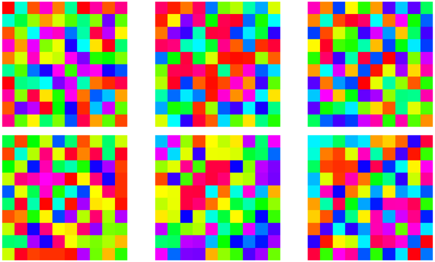
RMT: The idea

The dynamical ensemble will be based on
Random Matrix Theory.

Study of particular systems

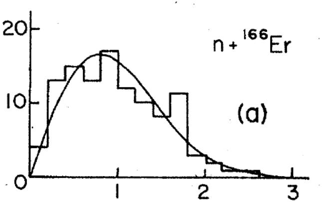


Ensemble of matrices with random entries

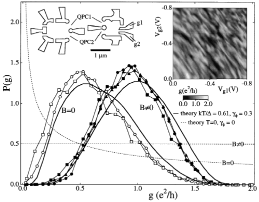


- An RMT study can predict generic properties
- It is based on unitary invariance and minimal information.

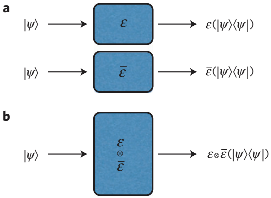
RMT: why do we use it (because it works!)



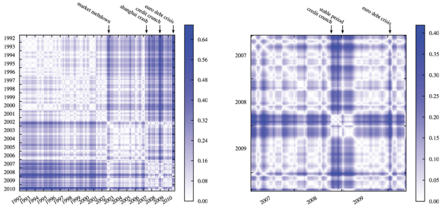
Nuclear physics
(Bohigas et. al. RMP, 1972)



Mesoscopic systems
(Huibers et. al. PRL, 1998)



Quantum information
(Hastings, Nat. Phys. 5, 2009)



Markets
(Seligman, among others, Scientific Reports 2012)

The program

Initial state:

$$|\psi(0)\rangle = |\text{Random}\rangle_{\text{env}} \otimes |\psi\rangle_{\text{central}}$$

Final state:

$$\rho_{\text{central}}(t) = \text{tr}_{\text{env}} |\psi(t)\rangle\langle\psi(t)|, \quad |\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\psi(0)\rangle$$

with H inspired in the classical ensembles, and t such that

$$P[\rho_{\text{central}}(t)] = P_{\text{target}}.$$

How are the eigenvalues of ρ_{central} distributed?
How does this distribution compare to the "static" distribution?

Cases

Four cases are studied:

- Global Hamiltonian
- Coupling Hamiltonian
- Spectator Hamiltonian
- Common environment Hamiltonian

Global Hamiltonian

The global Hamiltonian is simply

$$H = H^{(\mathrm{GUE})}_{\mathrm{env,central}}$$

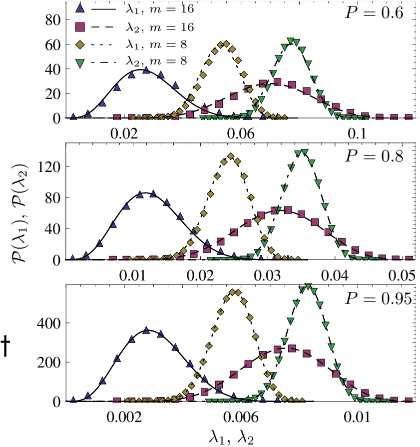
- Simple, conceptually
- Analytically tractable
- Already some crude approximation of coupling with environment

Global Hamiltonian

The global Hamiltonian is simply

$$H = H^{(\mathrm{GUE})}_{\mathrm{env,central}}$$

- Simple, conceptually
- Analytically tractable
- Already some crude approximation of coupling with environment



It is a good approximation. Almost perfect, for large P

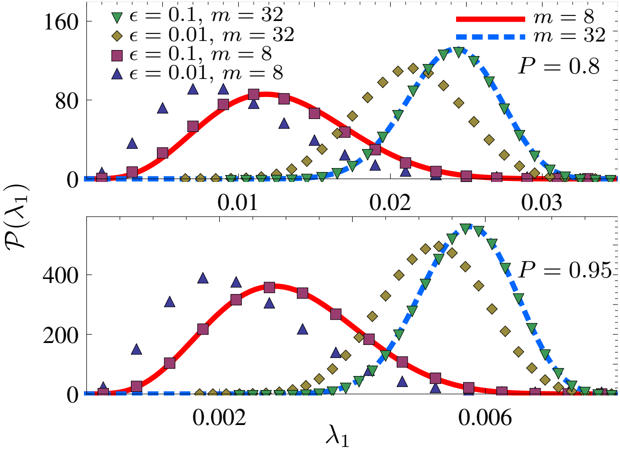
Coupling Hamiltonian

This Hamiltonian, has the notion of a separate part of the Hilbert space $\mathcal{H}_{\text{central}} \otimes \mathcal{H}_{\text{env}}$:

$$H = H_{\text{env}}^{(\text{GUE})} + \epsilon V_{\text{env,central}}^{(\text{GUE})}$$

- More realistic, closer to the experiment
- Different regimes, from weak to strong coupling
- Reduces to the previous one for $\epsilon \to \infty$

Coupling Hamiltonian (2)



Marginal distribution of λ_1 for two *P*s and two ϵs. For small ϵ, we find deviations, but already for moderate and large ϵ, the ansatz is very good.

Coupling a part of the system

What happens if only a part of the system is coupled?

- Spectator Hamiltonian

  $$H = H_{\text{env}}^{(\text{GUE})} + \epsilon V_{\text{env},\mathrm{q}_1}^{(\text{GUE})}$$

  simplest to study decoherence in 2 qubit systems. No dynamics on the second qubit.
- Common environment Hamiltonian

  $$H = H_{\text{env}}^{(\text{GUE})} + \epsilon V_{\text{env},\mathrm{q}_1}^{(\text{GUE})} + \epsilon V_{\text{env},\mathrm{q}_2}^{(\text{GUE})}.$$

  Closer to an experimental realization.

Added player: entanglement.

Initially low entangled state

For an initially weakly entangled states, we have in the spectator model:

$$\rho_{1,2}(t) = \rho_1(t) \otimes |\psi\rangle_2\langle\psi|_2$$

(no freedom) and in the common environment case

$$\rho_{1,2}(t) \approx \rho_1(t) \otimes \rho_2(t).$$

(one degree of freedom).

Initially low entangled state

For an initially weakly entangled states, we have in the spectator model:

$$\rho_{1,2}(t) = \rho_1(t) \otimes |\psi\rangle_2\langle\psi|_2$$

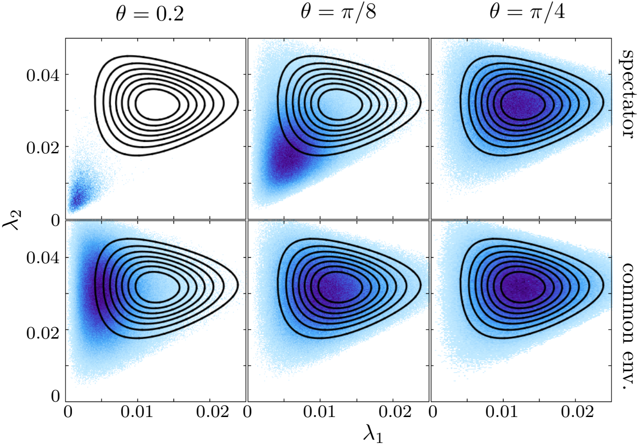
(no freedom) and in the common environment case

$$\rho_{1,2}(t) \approx \rho_1(t) \otimes \rho_2(t).$$

(one degree of freedom).

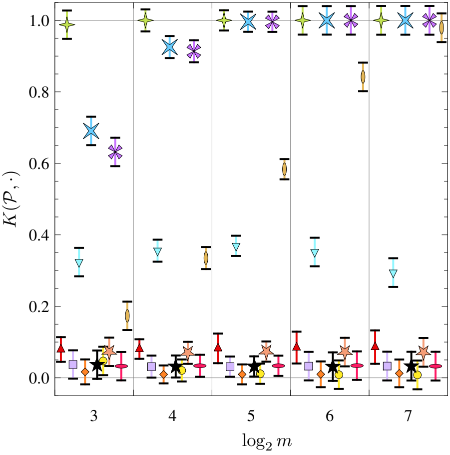
What about initially entangled states?

Behavior of entangled states (distribution of $\lambda_{1,2}$)



- $P = 0.8$, $m = 8$, $|\psi(0)\rangle = \sin\theta|00\rangle + \cos\theta|11\rangle$
- Spectator origin → ok
- Common axis → ok

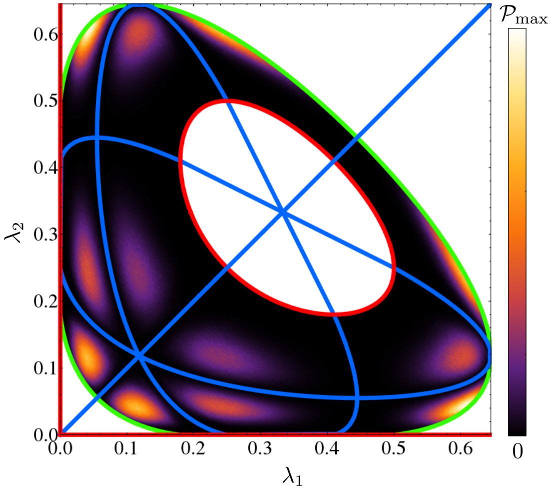
Summary of all comparisons



Symbol	Hamiltonian	P	ϵ	θ
▲	Global Hamiltonian	0.6		
■	Global Hamiltonian	0.8		
◆	Global Hamiltonian	0.95		
▽	Coupling Hamiltonian	0.8	0.01	
●	Coupling Hamiltonian	0.8	0.1	
★	Coupling Hamiltonian	0.8	1	
✦	Spectator Hamiltonian	0.8	0.1	0.2
✕	Spectator Hamiltonian	0.8	0.1	$\pi/8$
★	Spectator Hamiltonian	0.8	0.1	$\pi/4$
✳	Common environment Ham.	0.8	0.1	0.2
⬮	Common environment Ham.	0.8	0.1	$\pi/8$
⬬	Common environment Ham.	0.8	0.1	$\pi/4$

Other convex functions

Distribution at fixed von Neumann entropy



Nodes and probability distribution of eigenvalues for $S = 1.5$.

Conclusions

- Vandermonde determinant crucial for the structure of the static ensemble, though not for integrating at large dimensions.
- Good agreement between the two ensembles proposed, if the coupling is not so small.
- It is crucial to have maximal entanglement to have good agreement between the ensembles.
- Seems to be the same for other Rényi entropies, and maybe also for other convex functions.